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EFFECT OF SHEAR DEFORMATION ON VIBRATION OF ANTISYMMETRIC ANGLE-PLY LAMINATED RECTANGULAR PLATES

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Abstract—The title problem is solved in closed form for the case of all edges simply supported. A displacement formulation of the heterogeneous shear-deformable plate theory originated by Yang, Norris and Stavsky is used. Material properties typical of a highly directional composite material (high-modulus graphite/epoxy) are used and numerical results are presented showing the parametric effects of aspect ratio, length/thickness ratio, number of layers, and lamination angle. The effects of deleting rotatory inertia and in-plane inertia, singly and in combination, were also investigated. The information presented should be useful to composite-structure designers, to researchers seeking to obtain better correlation between theory and experiment, and to numerical analysts in checking out finite-element programs.

NOTATION

A_{ij}, B_{ij}, D_{ij} extensional, flexural-extensional coupling, and flexural stiffnesses
a, b plate planform dimensions in x, y directions
C indicator for including ($C = 1$) or omitting ($C = 0$) in-plane inertias
C_{kl} element of the frequency determinant, defined in eqns (10)
E_L, E_T ply material elastic moduli in directions along fibers and normal to them, respectively
G_{LT}, G_{LZ}, G_{TZ} ply material in-plane and thickness shear moduli
h total thickness of plate
I laminate rotatory inertia coefficient per unit midplane area
k_4, k_5 shear correction coefficients associated with the yz and xz planes, respectively
L_{kl} element of the operator matrix, defined in eqn (5)
M_i, N_i stress couple and stress resultant, respectively
P laminate normal inertia coefficient per unit midplane area
Q_i shear stress resultants
Q_{ij} plane-stress reduced stiffness coefficients
t time
U, V, W amplitudes of u, v, w
u, v, w displacement components in x, y, z directions
X, Y amplitudes of $h\psi_x, h\psi_y$
x, y, z position coordinates in longitudinal, transverse, and thickness directions
α, β $m\pi/a$, $n\pi/b$
ρ density
ψ_x, ψ_y slopes in xz and yz planes
ω frequency association with m and n half waves, rad/sec
$(\ )_{,f}$ partial differentiation with respect to $f (f = x, y, t)$

INTRODUCTION

It has long been known that due to their relatively low transverse shear modulus, filamentary composite materials exhibit much larger thickness shear (transverse shear) effects than do beams or plates having the same geometry but made of homogeneous, isotropic materials [1–3].

A great variety of shear deformation theories have been proposed and are reviewed in [4]. They range from the first such theory by Stavsky [5] for laminated isotropic plates, through the theory of Yang, Norris and Stavsky [6] for laminated anisotropic plates to various effective stiffness theories such as those discussed by Sun and Whitney [7], Whitney and Sun's higher order theory [8], and the three-dimensional elasticity-theory approach of Srinivas and Rao [9]. It has been shown by various investigators [7, 9] that the YNS (Yang–Norris–Stavsky) theory is adequate for predicting gross structural behavior in the first few "flexural" modes but not the higher "shear" modes.

The first extensive application of the YNS theory to commonly encountered laminate configurations was due to Whitney and Pagano [10], who considered cylindrical bending of antisymmetric cross-ply and angle-ply plate strips under sinusoidal load distribution and free vibration of antisymmetric angle-ply plate strips. Fortier and Rossettos[11] analyzed free vibration of thick rectangular plates of unsymmetric cross-ply construction while Sinha and Rath[12] considered both vibration and buckling for the same type of plates. Noor[13] treated buckling of thick rectangular plates of both symmetric and antisymmetric construction.

Apparently the free vibration of finite-dimension angle-ply plates with shear deformation has been neglected. The purpose of the present analysis is to provide a closed-form solution to this problem for the case of a certain class of hinged edge supports. The results should not only be of interest in their own right but also as a check for finite-element algorithms.

2. GOVERNING EQUATIONS

The plate is assumed to consist of an even number of identical orthotropic layers oriented alternately at angles θ and $-\theta$. Then the equations of motion for the YNS theory reduce to the following:

$$N_{1,x} + N_{6,y} = Cpu_{,tt} \tag{1a}$$

$$N_{6,x} + N_{2,y} = Cpv_{,tt} \tag{1b}$$

$$Q_{x,x} + Q_{y,y} = pw_{,tt} \tag{1c}$$

$$M_{1,x} + M_{6,y} - Q_x = I\psi_{x,tt} \tag{1d}$$

$$M_{6,x} + M_{2,y} - Q_y = I\psi_{y,tt} \tag{1e}$$

where x, y are midplane position coordinates parallel to the respective plate edges; u, v, w are the displacement components in the x, y, z (thickness) directions; ψ_x and ψ_y are the shear rotations; M_i and N_i are the stress couples and in-plane stress resultants; Q_i are the thickness shear stress resultants; t is time; $(\ )_{,x} \equiv \partial(\ )/\partial x$; C is an in-plane inertia indicator; and p and I are the normal and rotatory inertia coefficients per unit midplane areas defined as follows:

$$(p, I) = \int_{-h/2}^{h/2} (1, z^2)\rho\, dz = (\rho h, \rho h^3/12). \tag{2}$$

Here h is the total laminate thickness and ρ is the material density.

It is noted that coupling inertia terms are omitted in eqns (1a), (1b), (1d) and (1e) since the density is the same for each layer.

For an antisymmetric angle-ply laminate, the laminate constitutive relations can be expressed as follows:

$$\begin{Bmatrix} N_1 \\ N_2 \\ N_6 \\ M_1 \\ M_2 \\ M_6 \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & 0 & 0 & 0 & B_{16} \\ A_{12} & A_{22} & 0 & 0 & 0 & B_{26} \\ 0 & 0 & A_{66} & B_{16} & B_{26} & 0 \\ 0 & 0 & B_{16} & D_{11} & D_{12} & 0 \\ 0 & 0 & B_{26} & D_{12} & D_{22} & 0 \\ B_{16} & B_{26} & 0 & 0 & 0 & D_{66} \end{bmatrix} \begin{Bmatrix} u_{,x} \\ v_{,y} \\ v_{,x} + u_{,y} \\ \psi_{x,x} \\ \psi_{y,y} \\ \psi_{y,x} + \psi_{x,y} \end{Bmatrix} \tag{3a}$$

$$\begin{Bmatrix} Q_y \\ Q_x \end{Bmatrix} = \begin{bmatrix} k_4^2 A_{44} & 0 \\ 0 & k_5^2 A_{55} \end{bmatrix} \begin{Bmatrix} w_{,y} + \psi_y \\ w_{,x} + \psi_x \end{Bmatrix} \tag{3b}$$

where the extensional, flexural-extensional coupling, and flexural or twisting stiffnesses are defined as follows:

$$(A_{ij}, B_{ij}, D_{ij}) \equiv \int_{-h/2}^{h/2} (1, z, z^2) Q_{ij}\, dz. \tag{3c}$$

Here k_4^2 and k_5^2 are shear correction coefficients which may be calculated by various static and dynamic methods[6, 8, 14, 15].

Substituting equations (3a–b) into equations (1a–e), we obtain the following set of differential equations in operator form:

$$[L_{kl}]\begin{Bmatrix} u \\ v \\ w \\ h\psi_y \\ h\psi_x \end{Bmatrix} = 0 \quad (k, l = 1, 2, 3, 4, 5) \tag{4}$$

where $[L_{kl}]$ is a symmetric linear differential operator matrix with the following components:

$$\begin{aligned}
L_{11} &\equiv A_{11}d_x^2 + A_{66}d_y^2 - Cpd_t^2 \\
L_{12} &\equiv (A_{12} + A_{66})d_xd_y \\
L_{13} &\equiv 0 \\
L_{14} &\equiv (B_{16}/h)d_x^2 + (B_{26}/h)d_y^2 \\
L_{15} &\equiv (2B_{16}/h)d_xd_y \\
L_{22} &\equiv A_{66}d_x^2 + A_{22}d_y^2 - Cpd_t^2 \\
L_{23} &\equiv 0 \\
L_{24} &\equiv (2B_{26}/h)d_xd_y \\
L_{25} &\equiv L_{14} \\
L_{33} &\equiv -k_5^2A_{55}d_x^2 - k_4^2A_{44}d_y^2 + pd_t^2 \\
L_{34} &\equiv -(k_4^2A_{44}/h)d_y \\
L_{35} &\equiv -(k_5^2A_{55}/h)d_x \\
L_{44} &\equiv (D_{66}/h^2)d_x^2 + (D_{22}/h^2)d_y^2 - (k_4^2A_{44}/h^2) - (I/h^2)d_t^2 \\
L_{45} &\equiv (D_{12} + D_{66})h^{-2}d_xd_y \\
L_{55} &\equiv (D_{11}/h^2)d_x^2 + (D_{yy}/h^2)d_y^2 - (k_5^2A_{55}/h^2) - (I/h^2)d_t^2
\end{aligned} \tag{5}$$

Here $d_x \equiv \partial(\,)/\partial x$, etc.

3. APPLICATION TO PLATE HINGED ON ALL EDGES

The following boundary conditions are considered:

$$\begin{aligned}
&u(0, y) = u(a, y) = 0 \quad && N_6(x, 0) = N_6(x, b) = 0 \\
&N_6(0, y) = N_6(a, y) = 0 \quad && v(x, 0) = v(x, b) = 0 \\
&w(0, y) = w(a, y) = 0 \quad && w(x, 0) = w(x, b) = 0 \\
&M_1(0, y) = M_1(a, y) = 0 \quad && M_2(x, 0) = M_2(x, b) = 0 \\
&\psi_y(0, y) = \psi_y(a, y) = 0 \quad && \psi_x(x, 0) = \psi_x(x, b) = 0.
\end{aligned} \tag{6}$$

The boundary conditions on u, v, w, M_1, M_2 and N_6 are the same ones considered by Whitney and Leissa[16] using the Reissner–Stavsky[17] classical theory of thin laminated plates. Of course, the ones on ψ_x and ψ_y are only present in a shear-deformation theory, such as the one considered here.

The governing eqns (4) and boundary conditions (6) are exactly satisfied in closed form by the following set of functions:

$$\begin{aligned}
u &= U \sin \alpha x \cos \beta y \cos \omega t \\
v &= V \cos \alpha x \sin \beta y \cos \omega t \\
w &= W \sin \alpha x \sin \beta y \cos \omega t \\
h\psi_y &= Y \sin \alpha x \cos \beta y \cos \omega t \\
h\psi_x &= X \cos \alpha x \sin \beta y \cos \omega t
\end{aligned} \tag{7}$$

where ω is the natural frequency associated with modes m, n and

$$\alpha \equiv m\pi/a; \quad \beta \equiv n\pi/b. \tag{8}$$

Here m and n are the modal wave numbers associated with directions x and y and a and b are the plate dimensions in directions x and y.

Substituting solutions (7) into governing eqns (4), one obtains the following 5×5 frequency determinantal equation:

$$|C_{kl}| = 0 \quad (k, l = 1, 2, 3, 4, 5) \tag{9}$$

where $|C_{kl}|$ is a symmetric determinant containing the following elements:

$$\begin{aligned}
C_{11} &= -A_{11}\alpha^2 - A_{66}\beta^2 + C\rho\omega^2 \\
C_{12} &= -(A_{12} + A_{66})\alpha\beta \\
C_{13} &= 0 \\
C_{14} &= -(B_{16}/h)\alpha^2 - (B_{26}/h)\beta^2 \\
C_{15} &= -(2B_{16}/h)\alpha\beta \\
C_{22} &= -A_{66}\alpha^2 - A_{22}\beta^2 + C\rho\omega^2 \\
C_{23} &= 0 \\
C_{24} &= -(2B_{26}/h)\alpha\beta \\
C_{25} &= C_{15} \\
C_{33} &= -k_5^2A_{55}\alpha^2 - k_4^2A_{44}\beta^2 + \rho\omega^2 \\
C_{34} &= -(k_4^2A_{44}/h)\beta \\
C_{35} &= -(k_5^2A_{55}/h)\alpha \\
C_{44} &= -(D_{66}/h^2)\alpha^2 - (D_{22}/h^2)\beta^2 - (k_4^2A_{44}/h^2) + (I/h^2)\omega^2 \\
C_{45} &= -(D_{12} + D_{66})h^{-2}\alpha\beta \\
C_{55} &= -(D_{11}/h^2)\alpha^2 - (D_{66}/h^2)\beta^2 - (k_5^2A_{55}/h^2) + (I/h^2)\omega^2.
\end{aligned} \tag{10}$$

4. NUMERICAL EXAMPLES

Calculations were first carried out for the fundamental natural frequency of a square plate consisting of four layers oriented at 45°, −45°, 45°, −45°, respectively. The dimensionless material properties used are the same as those used by Whitney and Pagano [10] and are typical of graphite-epoxy:

$$E_L/E_T = 40, \quad G_{LT}/E_T = G_{LZ}/E_T = 0.6, \quad G_{TZ}/E_T = 0.5, \quad \nu_{LT} = 0.25.$$

A value of 5/6 was used for the shear coefficients k_4^2 and k_5^2.

For the convenience of designers in a wide variety of design situations, the fundamental-frequency results are presented here in dimensionless form as a function of the side-to-thickness ratio (a/h): Fig. 1† is for the case of a square plate $(a/b = 1)$; Fig. 2 is applicable to a plate strip $(a/b \to 0)$. For comparison both the results obtained by the present shear deformation theory (labeled SDT) and those obtained by classical thin plate theory (labeled CPT) are shown.

It was demonstrated by Wu and Vinson [19], for example, that the effect of a/h in reducing

†The curve in Fig. 1 does not agree with that given as Fig. 5 in the work of Whitney and Pagano [10]. However, their figure can be shown to be incorrect in that the classical-plate-theory (CPT) solution given in [10], $\omega a^2(\rho/E_T h^3)^{1/2} \approx 14.5$, is not correct. The correct value, as given by Jones, Morgan and Whitney [18], is $\omega a^2(\rho/E_T h^3)^{1/2} = 23.53$, which is in excellent agreement with the present value of 23.53.

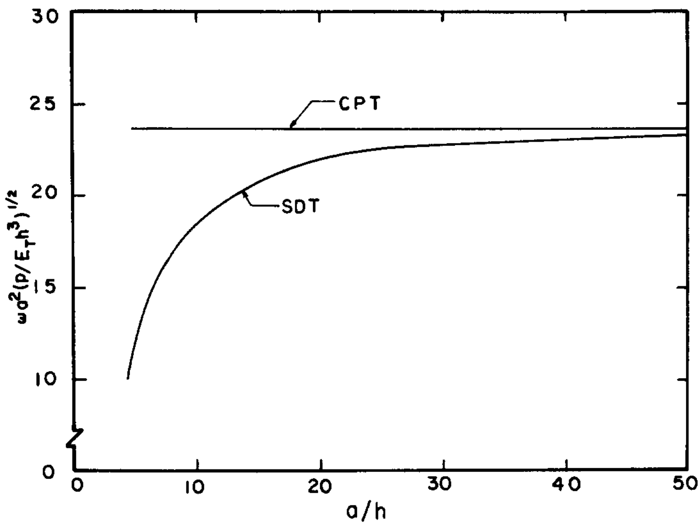

Fig. 1. Fundamental vibration frequency for a four-layer antisymmetric angle-ply square plate of graphite-epoxy composite material by classical plate theory (CPT) and shear deformation theory (SDT).

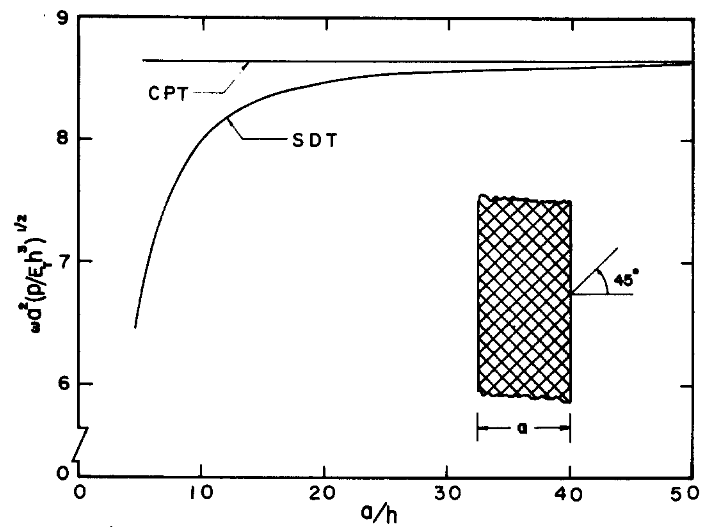

Fig. 2. Comparison of fundamental vibration frequency for a four-layer antisymmetric angle-ply plate strip of graphite-epoxy composite material by CPT and SDT.

the fundamental frequency is much more pronounced for plates of unidirectional advanced composite material than it is for homogeneous, isotropic plates of the same planform dimensions. The explanation for this is the low ratio of transverse shear moduli to inplane Young's moduli for the advanced composite. This same trend was found in the present investigation. For example, for a square plate with $a/h = 10$, the SDT prediction of the fundamental frequency is 3.43% lower than the CPT one for a homogeneous, isotropic plate but 28.3% lower for the present angle-ply graphite/epoxy plate.

To facilitate extrapolation to aspect ratios (a/b) other than one or infinity, Fig. 3 presents dimensionless frequency as a function of a/b for various values of a/h.

The effect of shear-coupled flexural-extensional coupling (B_{16} and B_{26} stiffnesses) decreases as the number of layers is increased, thus allowing the dimensionless frequency factor to increase, as shown in Fig. 4. Also it is seen that the effect of increasing lamination angle, up to a value of 45°, is to increase the frequency (except for the two-layer case) as would be expected. However, it is noted that the effect of the number of layers is most pronounced at $\theta = 45°$, i.e. going from two to many layers increases the frequency by 33% at $\theta = 45°$ but only by 5.0% at $\theta = 5°$. This is consistent with the greater discrepancy between classical plate buckling and experiment at $\theta = 45°$ as reported by Ashton and Whitney[20].

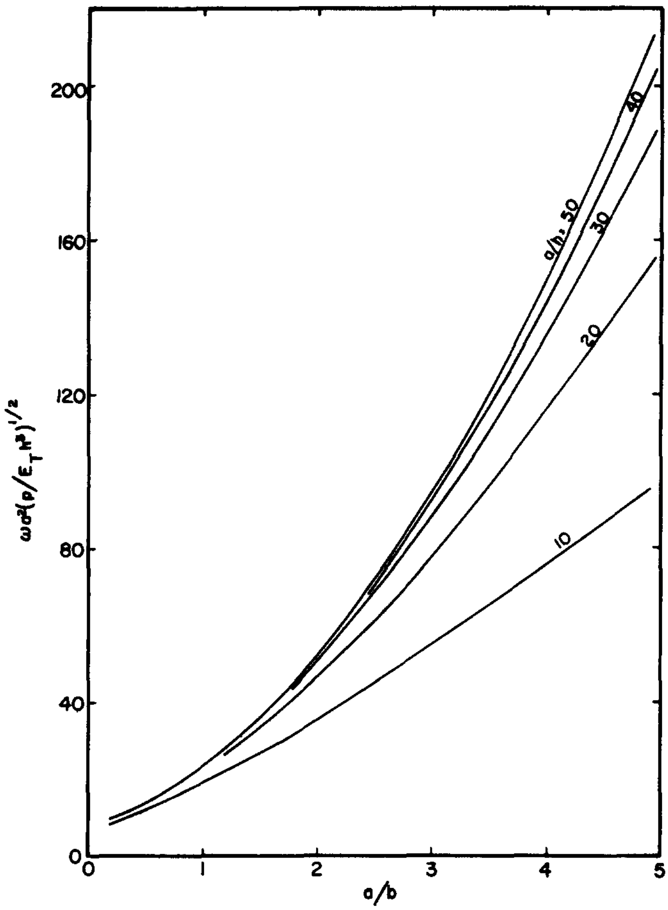

Fig. 3. Effects of plate aspect ratio (a/b) and length/thickness ratio (a/b) on the dimensionless fundamental frequency, $\omega a^2(\rho/E_T h^3)^{1/2}$, of a simply supported rectangular plate with high-modulus graphite/epoxy layers arranged at 45°, −45°, 45°, −45°.

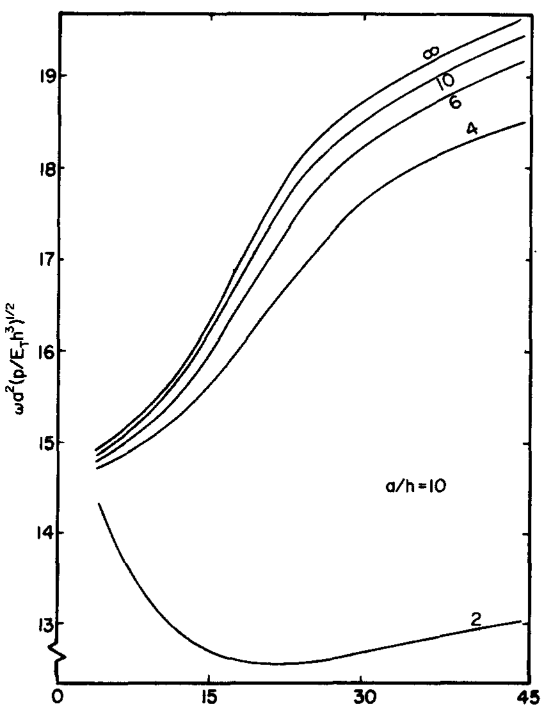

Fig. 4. Effects of lamination angle (θ) and number of layers on the dimensionless fundamental frequency $\omega a^2(\rho/E_T h^3)^{1/2}$ of a simply supported square plate laminated of high-modulus graphite/epoxy $(\theta, -\theta, \theta, -\theta;$ $a/h = 10)$.

In obtaining all of the above results, it was assumed that the shear factors k_4^2 and k_5^2 are equal to 5/6. If one does not consider a correction for the transverse shear distribution through the thickness, a value of 1.0 is tacitly assumed. However, various methods of obtaining the shear factors result in values ranging as low as 0.70. The effect of using different values of $k_4^2 = k_5^2$ on the dimensionless frequency of a four-ply 45° antisymmetric angle-ply plate is depicted in Fig. 5. As would be expected, the effect of increasing k_4^2 is to increase the dimensionless frequency and the effect is relatively more pronounced at low values of a/h.

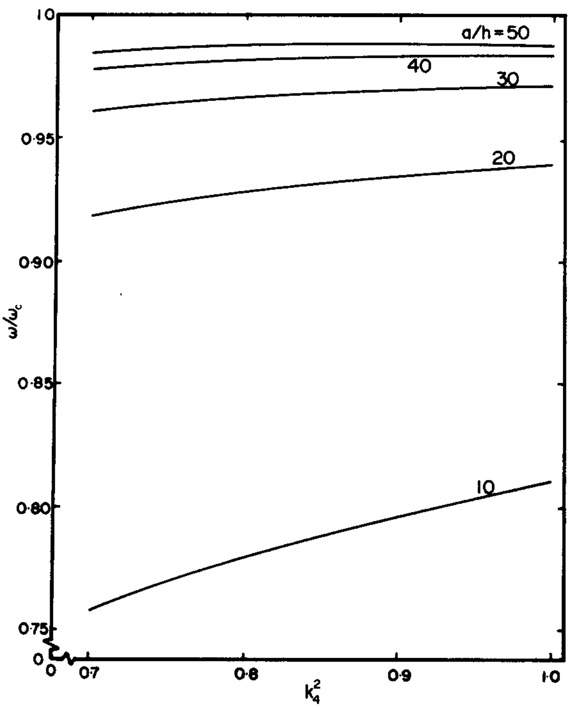

Fig. 5. Effect of shear factor $k_4^2 = k_5^2$ on relative fundamental frequency (ω/ω_c) for a square plate of high-modulus graphite/epoxy (45°, −45°, 45°, −45°).

In the past, certain investigators have chosen to neglect the effects of the secondary inertias, namely the in-plane inertias and the rotatory inertia. The effects of neglecting these parameters, either singly or both, are shown in Table 1. The neglect of these parameters, either singly or combined, introduces very little error in prediction of the fundamental frequency as can be seen in the table.

Table 1. Effects of in-plane inertia (Cp) and rotatory inertia (I) on the dimensionless fundamental frequency, $\omega a^2(\rho/E_T h^3)^{1/2}$, for $a/b = 1$, $a/h = 10$, 4 layers*

	θ, degrees			
Case	5	15	30	45
C≠0,I≠0	14.74	15.61	17.63	18.46
C=0,I≠0	14.76	15.63	17.65	18.47
C≠0,I=0	14.82	15.68	17.70	18.52
C=0,I=0	14.83	15.71	17.71	18.53

*Results are symmetric about θ=45°.

A comparison of the effects of both the longitudinal and transverse wave numbers (m and n) on the associated frequencies, as predicted by the present shear-theory analysis (SDT) and classical plate theory (CPT) is made in Table 2. Just as in the cases of isotropic plates and cross-ply plates [9], it is seen that the difference between the predictions of the two theories increases with increasing m and n.

Table 2. Dimensionless fundamental frequencies $\omega a^2(\rho/E_T h^3)^{1/2}$ for various longitudinal and transverse wave numbers (m and n) of a simply supported square plate with a length-to-thickness ratio of 10 and having high-modulus graphite/epoxy layers arranged at angles of 45°, −45°, 45° and −45°

Transverse Wave No.	Longitudinal Wave Number m				
n	1	2	3	4	5
1	18.46	34.87	54.27	75.58	97.56
	(23.53)	(53.74)	(98.87)	(160.35)	(238.72)
2	34.87	50.52	67.17	85.27	104.95
	(53.74)	(94.11)	(147.65)	(214.97)	(297.30)
3	54.27	67.17	82.84	99.02	116.28
	(98.87)	(147.65)	(211.75)	(288.76)	(379.12)
4	75.58	85.27	99.02	114.45	130.31
	(160.35)	(214.97)	(288.76)	(376.44)	(476.96)
5	97.56	104.95	116.28	130.31	145.57
	(238.72)	(297.30)	(379.12)	(476.96)	(588.20)

NOTE: Values in parentheses are dimensionless fundamental frequencies obtained by using CPT.

5. CONCLUSIONS

Based on the results discussed above, the following conclusions regarding antisymmetric angle-ply plates were reached:

1. The relative effect of transverse shear deformation on the fundamental frequency is greater for antisymmetric angle-ply plates than for homogeneous, isotropic plates of the same dimensions.

2. The effect of plate aspect ratio on the fundamental frequency is more pronounced in thicker plates (low a/h ratio) than it is for thin plates (high a/h ratio).

3. The effects of increasing angle θ (up to 45°) is to increase the fundamental frequency, except for the case of two layers for which it decreases.

4. Increasing the number of layers without changing the total thickness decreases the flexural-extensional coupling effect and thus increases the fundamental frequency. This effect of the number of layers is most pronounced at 45°.

5. The fundamental frequency is not affected very much when the shear factor is changed over the range 0.70–1.00.

6. Both the rotatory and in-plane inertias have little effect on the fundamental frequency even for a relatively thick plate ($a/h = 10$).

7. The error in CPT as compared with SDT increases very severely with an increase in either the longitudinal or transverse wave numbers.

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